Riemann Sums and Definite Integrals

-What happens if the intervals aren't even? A big rectangle here, a smaller rectangle there could still work.

- Does it matter, given the amount of rectangles we are using?

-The "long-way" of finding the area under the curve is known as a **Riemann Sum**.

-Consider the case where the number of rectangles increases and the width of the rectangle decreases.

As the number of rectangles increase, we say that the **norm** of the partition (or the width of the largest subinterval) decreases.

As $n \to \infty, \|\Delta\| \to 0$

This makes another version of our limit-sum definition:

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f\left(c_i\right) \Delta x_i$$

Definite Integrals

If f is defined on the closed interval $\left[a, b\right]$ and the limit

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f\left(c_i\right) \Delta x_i$$

exists, then f is **integrable** on $\left[a, b\right]$ and the limit is denoted by

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f\left(c_i\right) \Delta x_i = \int_a^b f\left(x\right) dx$$

The limit is called the **definite integral** of f from a to b. The number a is the **lower limit** of integration, and the number b is the **upper limit** of integration.

Continuity Implies Integrability

If a function f is continuous on the closed interval $\left[a,b\right]$, then f is integrable on $\left[a,b\right]$.

Example

Evaluate the definite integral $\int_{-2}^{1} 2x\,dx$

$$\Delta x = \frac{b-a}{n} = \frac{3}{n}$$

$$c_i = a + i\Delta x = -2 + \frac{3i}{n}$$

$$\int_{-2}^{1} 2x\,dx = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f\left(c_i\right)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} 2\left(-2 + \frac{3i}{n}\right)\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty} \frac{6}{n} \sum_{i=1}^{n} \left(-2 + \frac{3i}{n}\right)$$

$$= \lim_{n\to\infty} \frac{6}{n} \left\{-2n + \frac{3}{n}\left[\frac{n\left(n+1\right)}{2}\right]\right\}$$

$$= \lim_{n\to\infty}\left(-12+9+\frac{9}{n}\right)$$

$$= -3$$

-This function wasn't non-negative so it muddles the true definition of area!!

Properties of Definite Integrals

a) $\int_a^b k \bullet f(x)\,dx = k\int_a^b f(x)\,dx$

b) $\int_a^b \left[f(x) \pm g(x)\right]dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Preservation of Inequality

If f is integrable and non-negative on $\left[a,b\right]$ then

$$0 \le \int_a^b f(x)\,dx$$

If f and g are integrable on the closed interval $\left[a,b\right]$ and $f(x) \le g(x)$ for every x in $\left[a,b\right]$ then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

The Definite Integral as the Area of a Region

If f is continuous and non-negative on the closed interval $\left[a,b\right]$, then the area of the region bounded by the graph of f, the x-axis, and the lines $x=a$ and $x=b$ is:

$$\text{Area}=\int_a^b f\left(x\right)\,dx$$

Properties of Definite Integrals

If f is defined at $x=a$, then we define $\int_a^a f\left(x\right)\,dx=0$

If f is integrable on $\left[a,b\right]$, then we define $\int_b^a f\left(x\right)\,dx=-\int_a^b f\left(x\right)\,dx$

If f is integrable on 3 closed intervals determined by a, b, and c, then

$$\int_a^b f\left(x\right)\,dx=\int_a^c f\left(x\right)\,dx+\int_c^b f\left(x\right)\,dx$$

Fundamental Theorem of Calculus

Consider the connection between the uses of differentiation and definite integration.

Slope	**Area**
$\frac{\Delta y}{\Delta x}$	$\Delta y \Delta x$

If a function f is continuous on $\left[a,b\right]$ and F is an antiderivative of f on $\left[a,b\right]$ then

$$\int_a^b f\left(x\right)\,dx = F\left(b\right)-F\left(a\right)$$

Notation

a) $\displaystyle\int_a^b f\left(x\right)\,dx = F\left(x\right)\Big|_a^b$

$$= F\left(b\right)-F\left(a\right)$$

b) $\displaystyle\int_a^b f\left(x\right)\,dx = \Big[F\left(x\right)+C\Big]_a^b$

$$= \left[F\left(b\right)+C\right]-\left[F\left(a\right)+C\right]$$

$$= F\left(b\right)-F\left(a\right)$$

‘So we don’t need the constant of integration for definite integrals!!

Example

Evaluate each definite integral.

$$\int_1^2 \left(x^2-3\right)dx$$

$$=\left[\frac{x^3}{3}-3x\right]_1^2=\left(\frac{8}{3}-6\right)-\left(\frac{1}{3}-3\right)=-\frac{2}{3}$$

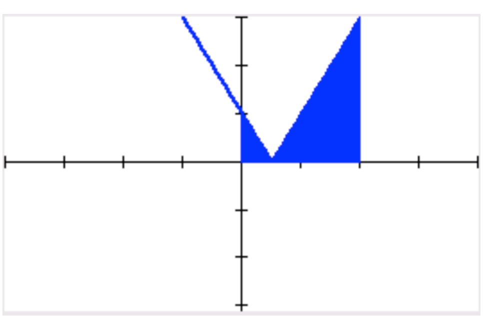
$$\int_1^4 3\sqrt{x}\,dx$$

$$=3\int_1^4 x^{1/2}dx=3\left[\frac{x^{3/2}}{3/2}\right]_1^4=2\left(4\right)^{3/2}-2\left(1\right)^{3/2}=14$$

$$\int_0^{\pi/4} \sec^2\left(x\right)dx$$

$$=\tan\left(x\right)\Big|_0^{\pi/4}=1-0=1$$

Example



$$y=\left|2x-1\right|$$

Evaluate $\int_0^2 \left|2x-1\right|dx$

$$|2x-1| = \begin{cases} -(2x-1) & x < 1/2 \\ 2x-1 & x \geq 1/2 \end{cases}$$

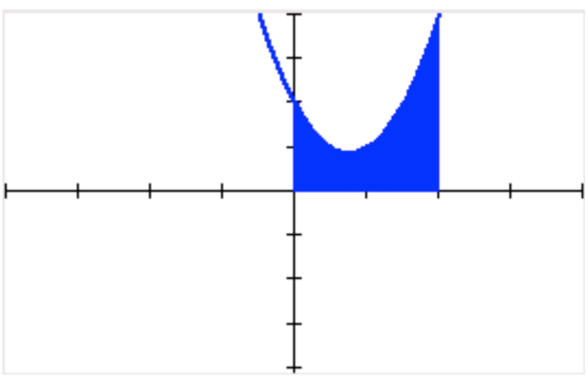
Rewrite:

$$\int_0^2 |2x-1|\, dx = \int_0^{1/2} -(2x+1)\, dx + \int_{1/2}^2 (2x-1)\, dx$$

$$= \left[-x^2 + x\right]_0^{1/2} + \left[x^2 - x\right]_{1/2}^2$$

$$= \left(-\frac{1}{4} + \frac{1}{2}\right) - (0+0) + (4-2) - \left(\frac{1}{2} - \frac{1}{2}\right) = \frac{5}{2}$$

Example



Find the area of the region bounded by the graph of $y = 2x^2 - 3x + 2$, the x-axis, and the vertical lines $x = 0$ and $x = 2$.

$$\text{Area} = \int_0^2 \left(2x^2 - 3x + 2\right) dx$$

$$= \left[\frac{2x^3}{3} - \frac{3x^2}{2} + 2x \right]_0^2$$

$$= \left(\frac{16}{3} - 6 + 4 \right) - (0 - 0 + 0)$$

$$= \frac{10}{3}$$